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Viscoelastic fingering with a pulsed pressure signal

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Abstract
We derive a generalized Darcy's law in the frequency domain for a linear viscoelastic fluid flowing in a Hele-Shaw cell. This leads to an analytic expression for the dynamic permeability that has maxima which are several orders of magnitude larger than the static permeability. We then follow an argument of de Gennes (1987 *Europhys. Lett.* **2** 195) to obtain the smallest possible finger width when viscoelasticity is important. Using this and a conservation law, we obtain the lowest bound for the width of a single finger displacing a viscoelastic fluid. When the driving force consists of a constant pressure gradient plus an oscillatory signal, our results indicate that the finger width varies in time following the frequency of the incident signal. Also, the amplitude of the finger width in time depends on the value of the dynamic permeability at the imposed frequency. When the finger is driven with a frequency that maximizes the permeability, variations in the amplitude are also maximized. This gives results that are very different for Newtonian and viscoelastic fluids. For the former ones the amplitude of the oscillation decays with frequency. For the latter ones on the other hand, the amplitude has maxima at the same frequencies that maximize the dynamic permeability.

The viscous fingering problem has been historically very important in the area of morphology of interfaces out of equilibrium [2] and has been a model system for describing displacement of viscous fluids in porous media. The viscous fingering problem is based on the study of the fluid interface in a two-phase flow confined in a Hele-Shaw cell [3] which consists of a pair of glass plates parallel to each other separated by a small gap. A viscous fluid occupies the space between the plates and it is pushed by a second fluid whose viscosity is relatively low. When the fluid is pushed laterally, the experiment is said to take place in a linear cell. The interface between the fluids is unstable and the structures that are formed when the interface destabilizes are called fingers. This is the so-called Saffman–Taylor instability [4]. Recently, viscous fingering experiments have been carried out with non-Newtonian fluids. Some of these experiments have used clays [5], polymer solutions [6, 7] and lyotropic lamellar phases [8].

Non-Newtonian fluids differ widely in their physical properties, with different fluids exhibiting a range of different properties, from plasticity and elasticity to shear thickening and shear thinning. Several studies have been made, both theoretically and experimentally, in order to differentiate between the effects caused by different properties [1, 7, 9, 10]. Nevertheless, there have been virtually no studies of the effect that a time varying driving force has on Saffman fingers [11, 12].

In this paper we analyse the effect of frequency on the width of a single finger displacing a viscoelastic fluid. To start, we derive a generalized Darcy's law in the frequency domain for a linear viscoelastic fluid flowing in a Hele-Shaw cell. This leads to an analytic expression for the dynamic permeability that has maxima which are several orders of magnitude larger than the static permeability. This is in agreement with results obtained for other geometries [13] that have been confirmed experimentally [14]. We then follow an argument of de Gennes [1] to obtain the smallest possible finger width when viscoelasticity is important. Using this, and a conservation law, we obtain the lowest bound for the width of a single finger displacing a viscoelastic fluid. When the driving force consists of a constant pressure gradient plus an oscillatory signal, our results indicate that the finger width varies in time following the frequency of the incident signal. Also the amplitude of the finger width in time depends on the dynamic permeability. This implies that when the finger is driven with a frequency that maximizes the permeability, variations in the amplitude are also maximized. For a fluid close to the Newtonian limit, the amplitude decays with frequency. For a viscoelastic fluid on the other hand, the amplitude will have maxima at the same frequencies that maximize the dynamic permeability.

We start our study by taking a Maxwell fluid, which is the simplest model of a linear viscoelastic fluid, and linearizing the equation governing the flow. In the frequency domain this equation is

$$-\rho\left(t_{\mathrm{r}}\omega^2+\mathrm{i}\omega\right)\hat{\vec{v}}-\eta\nabla^2\hat{\vec{v}}=-\left(1-\mathrm{i}\omega t_{\mathrm{r}}\right)\nabla\hat{p}. \tag{1}$$

Here both the velocity $\hat{\vec{v}}$ and the pressure $\hat{p}$ are in the frequency domain; that is, they are functions of space and frequency. t_{r}, η and ρ are respectively the relaxation time of the Maxwell fluid and the viscosity and the density of the fluid. We solve (1) for a homogeneous fluid flow in the x direction, confined between parallel plates at $z=\pm l$ subject to the boundary conditions $v_x(\pm l)=0$. We obtain the velocity profile between the plates. In order to obtain a generalized Darcy's law, we average over the z direction and obtain for the average flow

$$\langle\hat{v}\rangle=-\frac{K(\omega)}{\eta}\nabla\hat{p}, \tag{2}$$

where $K(\omega)$ is the dynamic permeability given by

$$K(\omega)=-\left(1-\frac{\tan\sqrt{\beta}l}{\sqrt{\beta}l}\right)\frac{(1-\mathrm{i}\omega t_{\mathrm{r}})}{\beta} \tag{3}$$

and $\beta(\omega)=\frac{\rho(t_{\mathrm{r}}\omega^2+\mathrm{i}\omega)}{\eta}$. This generalized Darcy's law is an equation in the frequency domain. We have verified that when the pressure gradient consists of a single Fourier mode for which $\omega\to 0$, we recover the steady state Darcy's law in the time domain. On the other hand, it is worth emphasizing that the limit $t_{\mathrm{r}}\to 0$ is the limit of a Newtonian fluid.

Figure 1 shows the real part of the normalized permeability $K(\omega)/K(0)$ versus the frequency for both a Maxwell fluid and a Newtonian fluid. The figure shows that for a Newtonian fluid the permeability is a monotonically decreasing function of frequency. For a Maxwell fluid on the other hand, there are frequencies for which there are resonances which give peaks for the permeability. For typical values of the density, viscosity, plate spacing and

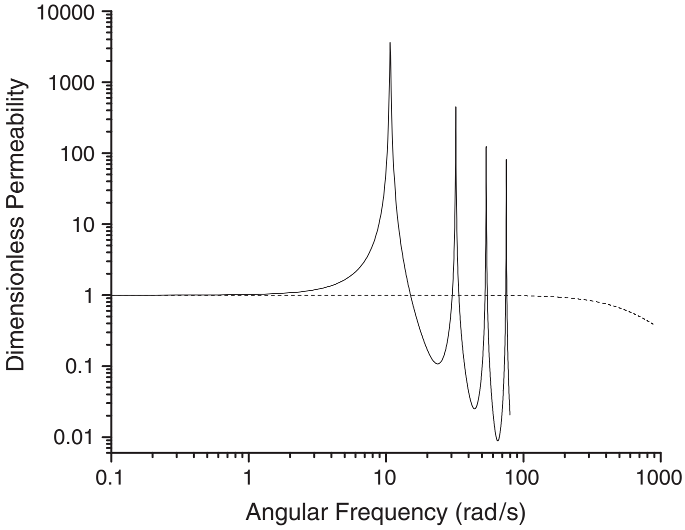

Figure 1. Dimensionless dynamic permeability versus frequency. The continuous curve is for a viscoelastic fluid with a relaxation time $t_r = 6$ s, the dotted curve for a Newtonian fluid. For both curves the viscosity $\eta = 0.7$ P, the density $\rho = 1$ g cm^{-3} and the spacing between the plates $L = 1$ mm.

relaxation time in the standard literature on Hele-Shaw problems, we find that this permeability can be two or three orders of magnitude larger than the Newtonian permeability; this is because the behaviour of the flow at such frequencies is dominated by the elastic properties of the fluid. What this result means is that if we displace the viscous fluid at the frequency that maximizes the permeability, the fluid will flow with the least possible resistance.

Figure 2 shows how the first maximum of the real part of the permeability shifts toward higher frequencies when the viscosity is increased. So, the more viscous the fluid, the higher the frequency that maximizes the dynamic permeability.

We now analyse the problem of a negligible viscosity fluid displacing a high viscosity viscoelastic fluid in a Hele-Shaw cell of width W—that is, the problem of the Saffman finger in the limit of infinite viscosity contrast. In particular, we analyse the case of a single finger propagating into the viscous fluid with a time dependent velocity $U(t)$. We call $\lambda(t)W$ the finger width. It is worth emphasizing that we are not considering a steady state. Both U and λ depend on time.

The quantity $U/(\lambda W)$ gives a characteristic frequency. The viscoelastic fluid has a characteristic time t_r which in the case of a Maxwell fluid is given by $t_r = \eta/G$, G being the rigidity modulus of the viscous fluid. When $U/(\lambda W) > 1/t_r$ the viscoelastic fluid behaves like a solid and there is no Saffman–Taylor instability. The allowed wavelengths should all correspond to $U/(\lambda W) < 1/t_r$ [1]. Therefore, the smallest possible finger width should be such that

$$\frac{U(t)}{\lambda(t)W} = \frac{1}{t_r}. \tag{4}$$

Conservation of matter implies that

$$U(t)\lambda(t) = V_\infty(t). \tag{5}$$

Here $V_\infty(t)$ is the fluid velocity very far from the finger tip. From equations (4) and (5) we can relate the finger width $\lambda(t)$ and the tip velocity $U(t)$ to the velocity at the extreme of the

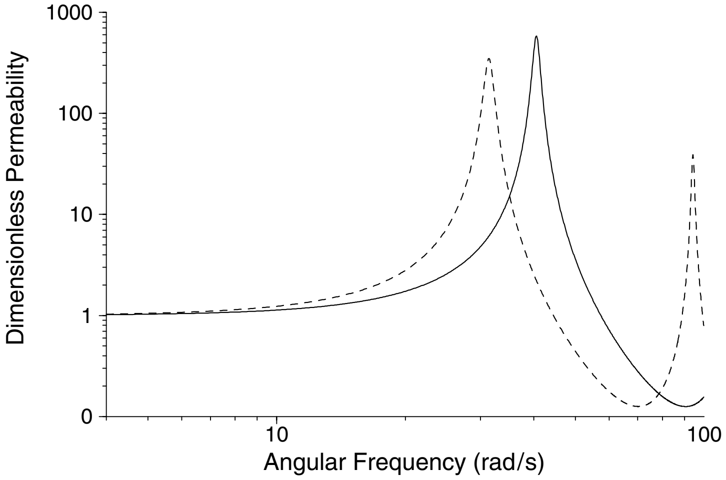

Figure 2. The effect of viscosity on the dimensionless dynamic permeability for a viscoelastic fluid. For the dotted curve the viscosity $\eta = 0.6$ P; for the continuous curve the viscosity $\eta = 1$ P. For both curves the density $\rho = 1$ g cm^{-3}, the spacing between the plates $L = 1$ mm and the relaxation time $t_r = 0.6$ s.

cell $V_\infty(t)$ as

$$\lambda^2(t) = \frac{t_r}{W} V_\infty(t) \tag{6}$$

and

$$U^2(t) = \frac{W}{t_r} V_\infty(t). \tag{7}$$

Experimentally, the parameter that can be controlled is the pressure difference at the extremes of the cell. So, given $\nabla p(t)$, we can make the following calculations:

$$\nabla p(t) \underset{\text{Fourier transform}}{\longrightarrow} \nabla \hat{p}(\omega),$$

$$\nabla \hat{p}(\omega) \underset{\text{Darcy's law in frequency domain}}{\longrightarrow} \hat{V}_\infty(\omega),$$

$$\hat{V}_\infty(\omega) \underset{\text{Inverse FT}}{\longrightarrow} V_\infty(t),$$

$$V_\infty(t) \longrightarrow \lambda(t), U(t).$$

In order for fingers to exist, the pressure gradient should be at any moment negative. Simple oscillatory signals are not possible since the instability will exist for only half of the period. So we are thinking, for example, of signals superimposed on pressure gradients that are large enough to destabilize the interface.

We consider the simple case of an oscillatory finger. Suppose we impose a pressure gradient of the form

$$\nabla p(t) = \nabla p_0 + \nabla p_a e^{-i\omega_0 t}, \tag{8}$$

that is, a constant pressure gradient ∇p_0 plus an oscillatory signal of frequency ω_0. ∇p_a is the amplitude of the oscillatory signal. We obtain the following expressions for the different steps presented above:

$$\nabla \hat{p}(\omega) = \sqrt{2\pi} \nabla p_0 \delta(\omega) + \sqrt{2\pi} \nabla p_a \delta(\omega - \omega_0) \tag{9}$$

$$\hat{V}_\infty(\omega) = -\frac{\sqrt{2\pi}}{\eta} K(\omega)\left[\nabla p_0 \delta(\omega) + \nabla p_{\rm a} \delta(\omega - \omega_0)\right] \tag{10}$$

$$V_\infty(t) = -\frac{\nabla p_0}{\eta} K(0) - \frac{\nabla p_{\rm a}}{\eta} K(\omega_0) {\rm e}^{-{\rm i}\omega_0 t}. \tag{11}$$

Now, for the steady state we know that

$$V_\infty^{\rm ss} = -\frac{\nabla p_0}{\eta} K(0) \tag{12}$$

and we can express the results as

$$V_\infty(t) = V_\infty^{\rm ss}\left[1 + \frac{\nabla p_{\rm a}}{\nabla p_0}\frac{K(\omega_0)}{K(0)} {\rm e}^{-{\rm i}\omega_0 t}\right], \tag{13}$$

$$\lambda^2(t) = \lambda_{\rm ss}^2\left[1 + \frac{\nabla p_{\rm a}}{\nabla p_0}\frac{K(\omega_0)}{K(0)} {\rm e}^{-{\rm i}\omega_0 t}\right], \tag{14}$$

$$U^2(t) = U_{\rm ss}^2\left[1 + \frac{\nabla p_{\rm a}}{\nabla p_0}\frac{K(\omega_0)}{K(0)} {\rm e}^{-{\rm i}\omega_0 t}\right]. \tag{15}$$

Here $\lambda_{\rm ss}$ and $U_{\rm ss}$ are defined in terms of $V_\infty^{\rm ss}$ through (6) and (7). What it is interesting to note is that the behaviour is totally different for viscous fluids in the Newtonian limit and for viscoelastic fluids. We focus our attention on the amplitude of the oscillatory term in equation (14). The same applies to equations (13) and (15). For fluids close to the Newtonian limit, the ratio $\frac{K(\omega_0)}{K(0)}$ decays with frequency. Also it is always smaller than or equal to one. This implies that variations in the width of the finger are small. On the other hand, for viscoelastic fluids, the ratio $\frac{K(\omega_0)}{K(0)}$ does not have a monotonic behaviour as a function of ω as can be seen from the figures. It can be several orders of magnitude larger than in the Newtonian case. Therefore, if the imposed signal has a frequency ω_0 for which the dynamic permeability has a maximum, the amplitude will also have a maximum and the time variations of the finger width will be very large.

A word of caution is needed, since when one does not consider surface tension the finger width and the velocity at the finger tip are not independent. It is worth noticing that since the surface tension has a stabilizing effect for small wavelengths any consideration of surface tension would give fingers wider than or equal in width to the lowest bound reported in the present work. In order to solve the finite surface tension problem, the method in [4, 15] should be considered.

Recently a phase field model has been developed to study the classical viscous fingering problem in the infinite viscosity contrast limit [16]—that is, the case when a zero-viscosity fluid pushes a high viscosity Newtonian fluid. Such a model allows for a numerical simulation of a finger driven by a time dependent pressure gradient [12]. Simulation results show that close to the finger tip, the finger width oscillates in time with a frequency that follows the frequency of the incident signal. This is in agreement with equation (14). Simulations also predict that the amplitude of the oscillation decays monotonically with frequency. This is in agreement with equation (14) when the Newtonian limit $t_{\rm r} \to 0$ is taken in equation (3). These results for the Newtonian case partially confirm the theory presented here.

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